

EXISTENCE AND STABILITY OF STANDING WAVES FOR NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS WITH HARTREE TYPE NONLINEARITY

DAN WU

ABSTRACT. In this paper, we consider the nonlinear fractional Schrödinger equations with Hartree type nonlinearity. We obtain the existence of standing waves by studying the related constrained minimization problems via applying the concentration-compactness principle. By symmetric decreasing rearrangements, we also show that the standing waves, up to a translations and phases, are positive symmetric nonincreasing functions. Moreover, we prove that the set of minimizers is a stable set for the initial value problem of the equations, that is, a solution whose initial data is near the set will remain near it for all time.

1. INTRODUCTION

We consider the following fractional nonlinear Schrödinger equation with Hartree type nonlinearity

$$i\psi_t + (-\Delta)^\alpha \psi - (|\cdot|^{-\gamma} * |\psi|^2)\psi = 0, \quad (1.1)$$

where $0 < \alpha < 1$, $0 < \gamma < 2\alpha$ and $\psi(x, t)$ is a complex-valued function on $\mathbb{R}^d \times \mathbb{R}$, $d \geq 2$. The fractional Laplacian $(-\Delta)^\alpha$ is a non-local operator defined as

$$\mathcal{F}[(-\Delta)^\alpha \psi](\xi) = |\xi|^{2\alpha} \mathcal{F}\psi(\xi), \quad (1.2)$$

where the Fourier transform is given by

$$\mathcal{F}\psi(\xi) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \psi(x) e^{-i\xi \cdot x} dx. \quad (1.3)$$

The fractional Schrödinger equation plays a significant role in the theory of fractional quantum mechanics. It was formulated by N. Laskin [14][15][16] as a result of extending the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths. The Lévy processes, occurring widely in physics, chemistry and biology, lead to equations with the fractional Laplacians which have been recently studied by [1][8][23]. When $\alpha = \frac{1}{2}$, NLS (1.1) can be used to describe the dynamics of pseudo-relativistic boson stars in the mean-field limit, see [7]. When $\alpha = 1$, the Lévy motion becomes Brownian motion and the fractional Schrödinger equation turns to be the well-known classical nonlinear Schrödinger equation which has been studied by many authors, see for instance [2][3][17][21].

2000 *Mathematics Subject Classification.* 35Q55.

Key words and phrases. Fractional nonlinear Schrödinger equation; Hartree; Standing wave; Stability; concentration-compactness.

Recently, the fractional nonlinear Schrödinger equations with power type nonlinearity have been studied by [9][10][11]. In this paper, we consider Hartree type nonlinearity. It has been showed in [4] that the equation (1.1) is locally well-posed in $H^\alpha(\mathbb{R}^d)$ and globally well-posed under some conditions. In view of the scaling invariance, we know that the equation (1.1) is mass-critical if $\gamma = 2\alpha$ and mass-subcritical if $\gamma < 2\alpha$. For mass-critical case $\gamma = 2\alpha$, [5] and [6] investigate the blowup phenomena of NLS (1.1) with radial data. The aim of this paper is to investigate existence and stability of standing waves of NLS (1.1) in mass-subcritical case.

A standing wave of NLS (1.1) means a solution of the special form $\psi(x, t) = e^{i\omega t}u(x)$, where $\omega \in \mathbb{R}$ is a frequency. In order to study the existence and stability of standing waves to NLS (1.1), we first look for (ω, u) satisfying the stationary equation

$$(-\Delta)^\alpha u - (|\cdot|^{-\gamma} * |u|^2)u = \omega u \quad \text{in } \mathbb{R}^d, \quad (1.4)$$

where $u(x)$ is complex-valued. For studying the existence of solutions to (1.4), by the variational method, we can consider the following constrained minimization problem:

$$E_q := \inf\{E(u); u \in H^\alpha(\mathbb{R}^d), M(u) = q\}, \quad (1.5)$$

where the mass is defined as

$$M(u) = \int_{\mathbb{R}^d} |u(x)|^2 dx, \quad (1.6)$$

and the energy is

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 dx - \frac{1}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u(x)|^2 |u(y)|^2 dx dy. \quad (1.7)$$

Remark 1.1. N. Laskin [16] showed the hermiticity of the fractional Schrödinger operator and established the conservation laws of the mass and the energy.

We will denote the set of minimizers of problem (1.5) by

$$G_q := \{u \in H^\alpha; E(u) = E_q, M(u) = q\}.$$

Let S denote the set of the symmetric decreasing functions in $H^\alpha(\mathbb{R}^d)$, that is,

$$S = \{u \in H^\alpha(\mathbb{R}^d); u \geq 0 \text{ and } u(x) \leq u(y) \text{ if } |x| \geq |y|\}, \quad (1.8)$$

and let

$$S' = \{u \in H^\alpha(\mathbb{R}^d); u(x-y) = v(y) \text{ a.e. for some } v \in S \text{ and } y \in \mathbb{R}^d\}, \quad (1.9)$$

the set of translates (a.e.) of functions in S . Two functions u and v in S' are said to be equicentered if $u(x-y) = v(y)$ a.e. for some $v \in S$ and $y \in \mathbb{R}^d$.

Throughout this paper, we always denote $\|\cdot\| = \|\cdot\|_{H^\alpha(\mathbb{R}^d)}$ and $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^d)}$ for simplicity.

Our main results in this paper are the following.

Theorem 1.2 (Existence of standing waves). *Let $d \geq 2$, $0 < \alpha < 1$, and $\gamma < 2\alpha$. If $\{u_n\}$ is a minimizing sequence of problem (1.5), then there exists a sequence $\{y_n\} \subset \mathbb{R}^d$ such*

that $\{u_n(\cdot - y_n)\}$ contains a convergent subsequence in $H^\alpha(\mathbb{R}^d)$. In particular, there exists a minimizer for problem (1.5), which implies G_q is not a empty set. Moreover, we have

$$\lim_{n \rightarrow \infty} \inf_{g \in G_q} \|u_n - g\| = 0. \quad (1.10)$$

Theorem 1.3. *The standing waves obtained in Theorem 1.2 satisfy the following properties:*

- (1) *The standing waves are continuous, in particular, $G_q \subset C^{[2\alpha], 2\alpha - [2\alpha]}(\mathbb{R}^d)$, where $[2\alpha]$ is the integer part of 2α ;*
- (2) *If $g \in G_q$, then $|g| \in G_q$ and $|g| > 0$ on \mathbb{R}^d ;*
- (3) *The standing waves are symmetric decreasing after modified translations and phases, that is, $G_q \subset \{u; e^{i\theta}u(x - y) = v(y) \text{ a.e. for some } v \in S, \theta \in \mathbb{R} \text{ and } y \in \mathbb{R}^d\}$.*

Theorem 1.4 ($H^\alpha(\mathbb{R}^d)$ -stable). *Under the assumptions of Theorem 1.2, the set G_q is $H^\alpha(\mathbb{R}^d)$ -stable with respect to NLS (1.1), that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $u \in C(\mathbb{R}, H^\alpha(\mathbb{R}^d))$ is a solution to NLS (1.1) with the initial data u_0 satisfying*

$$\inf_{g \in G_q} \|u_0 - g\| < \delta,$$

then for all $t > 0$, we have

$$\inf_{g \in G_q} \|u(\cdot, t) - g\| < \varepsilon.$$

2. PRELIMINARIES

In this section, we will collect some results known in existing literature, which will be used in our paper. To start with, we recall the definition of $H^\alpha(\mathbb{R}^d)$, which is the fractional order Sobolev space defined as

$$H_p^\alpha(\mathbb{R}^d) := \{u : \mathbb{R}^d \rightarrow \mathbb{C}; u \in L^p \text{ and } \mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{\alpha}{2}} \mathcal{F}u] \in L^p\},$$

whose norm is given by

$$\|\cdot\|_{\alpha, p} = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{\alpha}{2}} \mathcal{F}u]\|_p.$$

In particular, we write $H^\alpha(\mathbb{R}^d) = H_2^\alpha(\mathbb{R}^d)$ for brevity. The following lemma gives an equivalent norm that is quite useful.

Lemma 2.1. *Let $0 < \alpha < 1$, the norm $\|\cdot\|_{\alpha, 2}$ of $H^\alpha(\mathbb{R}^d)$ is equivalent to*

$$\|\cdot\| = \|\mathcal{F}^{-1}[(1 + |\xi|^\alpha) \mathcal{F}\cdot]\|_2 = \|\cdot\|_2 + \|\cdot\|_{\dot{H}^\alpha(\mathbb{R}^d)}.$$

This result follows easily from the fundamental inequality

$$1 + |\xi|^\alpha \leq (1 + |\xi|^2)^{\frac{\alpha}{2}} \leq C(1 + |\xi|^\alpha),$$

and the definitions of $\|\cdot\|_{\alpha, 2}$ and $\|\cdot\|$.

Next, we give a lemma, which is another definition of the fractional Laplacian and will be frequently used later.

Lemma 2.2. *Let $0 < \alpha < 1$, and $u(x)$ be a function in the Schwartz class on \mathbb{R}^d , then the fractional Laplacian of u has a pointwise expression as*

$$(-\Delta)^\alpha u(x) = C_{d,\alpha} \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2\alpha}} dy,$$

where P.V. means the Cauchy principal value on the integral and $C_{d,\alpha}$ is some positive normalization constant.

The equivalence of two definitions of the fractional Laplacian can be proved by Riesz potential and the Green's second identity. Here we omit the details. [23] gives a simple proof.

The following inequality, which is due to G. H. Hardy, will play a major role in the nonlinearity estimates.

Lemma 2.3 (The Hardy's inequality). *For $0 < \gamma < d$, we have*

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x - y|^\gamma} dx \leq C \|u\|^2,$$

where the constant C depends on d and γ .

The following commutator estimates was developed in [10] using Kato and Ponce's result [13].

Lemma 2.4 (Commutator estimates). *If $0 < \alpha < 1$ and $f, g \in \mathcal{S}$, the Schwartz class, then the following holds:*

$$\|(-\Delta)^{\frac{\alpha}{2}}(fg) - f(-\Delta)^{\frac{\alpha}{2}}g\|_2 \leq C(\|\nabla f\|_{p_1}\|(-\Delta)^{\frac{\alpha-1}{2}}\|_{q_1} + \|(-\Delta)^{\frac{\alpha}{2}}f\|_{p_2}\|g\|_{q_2}),$$

where $q_1, p_2 \in [2, +\infty)$ and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2}$.

Lemma 2.5 (Fractional Rellich Compactness theorem). *Let $0 < \alpha < d$, $1 \leq p < \frac{d}{\alpha}$, $1 \leq q < \frac{dp}{d-\alpha p}$ and Ω is a bounded open set with smooth boundary. Suppose $\{u_n\}$ is a sequence in $L^p(\mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{R}^d} |(-\Delta + 1)^{\frac{\alpha}{2}} u_n(x)|^p dx$$

are uniformly bounded, then $\{u_n\}$ has a convergent subsequence in $L^q(\Omega)$.

Lemma 2.5 can be found in H. Hajaiej [12].

For any given Borel set A with finite Lebesgue measure, we define its symmetric rearrangement by

$$A^* = \{x; |x| < r\} \text{ with } \frac{|\mathbb{S}^{d-1}|}{d} r^d = \mathfrak{L}^d(A), \quad (2.1)$$

where $|\mathbb{S}^{d-1}|$ is the surface area of the unit ball in \mathbb{R}^d . This allowed us to define the symmetric decreasing rearrangement of a characteristic function of a set A as

$$\chi_A^*(x) = \chi_{A^*}(x) \text{ for } x \in \mathbb{R}^d. \quad (2.2)$$

Clearly $\chi_A^* \in S$ and $\|\chi_A^*\|_1 = \|\chi_{A^*}\|_1 = \mathfrak{L}^d(A)$. Given $f \in H^\alpha(\mathbb{R}^d)$, we define

$$f^*(x) = \int_0^\infty \chi_{\{|f|>t\}}^*(x) dt, \quad (2.3)$$

which has following properties:

- f^* is radial, nonnegative and nonincreasing, i.e. $f^* \in S$;
- If $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$, then

$$\|f^*\|_p = \|f\|_p. \quad (2.4)$$

Moreover, from Theorem 2.1. in [12] we have

Lemma 2.6. *Let $0 \leq \alpha \leq 1$, then we have*

$$\|u^*(x)\|_{\dot{H}^\alpha(\mathbb{R}^d)} \leq \|u(x)\|_{\dot{H}^\alpha(\mathbb{R}^d)}. \quad (2.5)$$

F. Riesz [22] showed the following inequality. For a recent account of the theorems, we refer the reader to [18].

Lemma 2.7 (Riesz's rearrangement inequality). *Let f, g and h be three nonnegative functions on \mathbb{R}^d , Then we have*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x)g(x-y)h(y)dx dy \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^*(x)g^*(x-y)h^*(y)dx dy. \quad (2.6)$$

Furthermore, E.H. Lieb [17] has established the strict version of the Riesz's rearrangement inequality.

Lemma 2.8 (Strict version of Lemma 2.7). *Under the assumptions of Lemma 2.7. If $g \in S$ and g is positive and strictly decreasing, that is,*

$$0 < g(x) < g(y) \text{ if } |x| > |y|, \quad (2.7)$$

then (2.6) is a strict inequality when the right hand side is finite unless f and h are equicentered functions in S' .

Theorem 2.9 (Global existence of weak solutions for NLS (1.1)). *If $0 < \alpha < 1$, $\gamma < 2\alpha$ and $\psi_0 \in H^\alpha(\mathbb{R}^d)$, then there exists a global weak solution $\psi(x, t) \in C(\mathbb{R}, H^\alpha(\mathbb{R}^d))$ to the Cauchy problem of nonlinear fractional Schrödinger equations (1.1) with the initial data $\psi(x, 0) = \psi_0(x)$.*

See [4] for more details.

3. THE PROOF OF MAIN RESULTS

In this section we give proofs of our main results listed in the first section. To begin with, we solve the constrained minimization problem (1.5). It is known that, in this kind of problem, the main difficulty concerns with the lack of compactness of the minimizing sequences $\{u_n\}$ for the problem. Indeed, two bad scenarios possible are

- Vanishing $u_n \rightharpoonup 0$,
- Dichotomy $u_n \rightharpoonup u$ and $\|u\|_2^2 \neq q$.

In order to rule out the above two cases and to show that the infimum is achieved, we employ the concentration-compactness principle developed by P.L. Lions. The best general reference about this method are [19] and [20]. First of all, we introduce the Lévy concentration function.

$$Q_n(r) := \sup_{y \in \mathbb{R}^d} \int_{B(y,r)} |u_n(x)|^2 dx.$$

Since $\{Q_n\}$ is locally of bounded total variation and uniformly bounded, by the Helly's selection theorem, we can find a convergent subsequence, denoted again by $\{Q_n\}$ such that there is a nondecreasing function $Q(r)$ satisfying

$$\lim_{n \rightarrow +\infty} Q_n(r) = Q(r), \text{ for all } r > 0.$$

Note that $0 \leq Q_n(r) \leq q$, there exists $\beta \in [0, q]$ such that

$$\lim_{r \rightarrow +\infty} Q(r) = \beta. \quad (3.1)$$

Lemma 3.1. *For every $q > 0$, we have $-\infty < E_q < 0$.*

Proof. For given $u \in H^\alpha(\mathbb{R})$ with $\|u\|_2 = q$, letting $u_\lambda = \lambda^{\frac{1}{2}} u(\lambda^{\frac{1}{d}} x)$, we then have $\|u_\lambda\|_2 = q$. By the definition of energy, we have

$$E(u_\lambda) = \frac{1}{2} \lambda^{\frac{2\alpha}{d}} \int_{\mathbb{R}^d} |(-\Delta)^\alpha u(x)|^2 dx - \frac{1}{4} \lambda^{\frac{\gamma}{d}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u(x)|^2 |u(y)|^2 dx dy.$$

Since $0 < \gamma < 2\alpha$, we can take $\lambda > 0$ sufficiently small such that $E(u_\lambda) < 0$. Hence $E_q < E(u_\lambda) < 0$.

On the other hand, Hardy's inequality implies

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u(x)|^2 |u(y)|^2 dx dy &\leq \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u(x)|^2 dx \|u\|_2^2 \\ &\leq C \|u\|_{\dot{H}^{\frac{\gamma}{2}}}^2 \|u\|_2^2. \end{aligned} \quad (3.2)$$

Using Sobolev's inequality and Young's inequality, we deduce that

$$\frac{1}{4} H_\gamma(u, u) \leq C \|u\|_{\dot{H}^\alpha}^{\frac{\gamma}{\alpha}} \|u\|_2^{4-\frac{\gamma}{\alpha}} \leq \varepsilon \|u\|_{\dot{H}^\alpha}^2 + C_\varepsilon \|u\|^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}}, \quad (3.3)$$

where ε is a sufficiently small positive constant. Hence, for $u \in H^\alpha(\mathbb{R}^d)$ with $\|u\|_2 = q$ and sufficiently small ε ,

$$E(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{2} q - \varepsilon \|u\|^2 - C_\varepsilon q^{\frac{4\alpha-\gamma}{2\alpha-\gamma}} \geq -\frac{1}{2} q - C_\varepsilon q^{\frac{4\alpha-\gamma}{2\alpha-\gamma}},$$

which implies $E_q > -\infty$. So, $-\infty < E_q < 0$. □

Lemma 3.2. *Vanishing does not occur, that is, $\beta > 0$, for every $q > 0$.*

To prove this lemma, we need the following two lemmas.

Lemma 3.3. *Every minimizing sequence $\{u_n\}$ for problem (1.5) is bounded in $H^\alpha(\mathbb{R}^d)$, and there exists a constant $\delta > 0$ such that $H_\gamma(u_n, u_n) \geq \delta > 0$ for sufficiently large n .*

Proof. Firstly, it follows from (3.3) in Lemma 3.1 that

$$\frac{1}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u_n(x)|^2 |u_n(y)|^2 dx dy \leq \varepsilon \|u_n\|^2 + C_\varepsilon \|u_n\|_2^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}}.$$

From this, we deduce that

$$\begin{aligned} \frac{1}{2} \|u_n\|^2 &\leq E(u_n) + \frac{1}{2} \|u_n\|_2^2 + \frac{1}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u_n(x)|^2 |u_n(y)|^2 dx dy \\ &\leq E(u_n) + \frac{1}{2} q + \varepsilon \|u_n\|^2 + C_\varepsilon q^{\frac{4\alpha-\gamma}{2\alpha-\gamma}} \end{aligned} \quad (3.4)$$

Since $\{u_n\}$ is a minimizing sequence, we can get the result by taking $\varepsilon < \frac{1}{2}$.

For the second part, suppose that the lemma were false. Then we could find subsequences $\{u_{n_k}\}$ such that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u_{n_k}(x)|^2 |u_{n_k}(y)|^2 dx dy \rightarrow 0, \text{ as } k \rightarrow +\infty.$$

By the definition of energy, it follows immediately that

$$E(u_{n_k}) \rightarrow E_q \geq 0, \text{ as } k \rightarrow +\infty,$$

which contradicts Lemma 3.1. \square

Lemma 3.4. Suppose $\{u_n\}$ is a minimizing sequence for the problem (1.5) and satisfying

$$\sup_{y \in \mathbb{R}^d} \int_{B(y,r)} |u_n(x)|^2 dx \rightarrow 0,$$

then, we have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u_n(x)|^2 |u_n(y)|^2 dx dy \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proof. Consider a minimizing sequence $\{u_n\}$ for problem (1.5). For every $\varepsilon > 0$, since $\{u_n\}$ are bounded in L^2 , we can find $r_\varepsilon > 0$ such that

$$\iint_{|x-y| \geq r_\varepsilon} \frac{1}{|x-y|^\gamma} |u_n(x)|^2 |u_n(y)|^2 dx dy \leq \frac{\varepsilon}{2}.$$

Next we divide the domain. For every positive r , we can find countable balls $\{B(z_i, r)\}$ such that

$$\mathbb{R}^d \subset \bigcup_{i=1}^{\infty} B(z_i, r),$$

and every point in \mathbb{R}^d belongs to at most $d+1$ of these balls, which implies

$$\sum_{i=1}^{\infty} \int_{B(z_i, r)} |u_n(x)|^2 dx \leq (d+1) \|u_n\|_2^2. \quad (3.5)$$

Consequently, if x in some $B(z_i, r)$ and $|x - y| \leq r_\varepsilon$, then there exists at most N_ε balls such that

$$\{y \in \mathbb{R}^d; |x - y| \leq r_\varepsilon, x \in B(z_i, r)\} \subset \bigcup_{k=1}^{N_\varepsilon} B(z_{i_k}, r),$$

where N_ε only depends on ε . By the above facts, using Hölder's and Hardy's inequalities, we have

$$\begin{aligned} & \iint_{|x-y| \leq r_\varepsilon} \frac{1}{|x-y|^\gamma} |u_n(x)|^2 |u_n(y)|^2 dx dy \\ & \leq \sum_{i=1}^{\infty} \int_{B_x(z_i, r)} \left[\sum_{k=1}^{N_\varepsilon} \int_{B_y(z_{i_k}, r)} \frac{1}{|x-y|^\gamma} |u_n(x)|^2 |u_n(y)|^2 dy \right] dx \\ & \leq \sum_{i=1}^{\infty} \|u_n(x)\|_{L^2(B_x(z_i, r))}^2 \left[\sum_{k=1}^{N_\varepsilon} \sup_{x \in B_x(z_i, r)} \int_{B_y(z_{i_k}, r)} \frac{1}{|x-y|^\gamma} |u_n(y)|^2 dy \right] \\ & \leq \sum_{i=1}^{\infty} \|u_n(x)\|_{L^2(B_x(z_i, r))}^2 \sum_{k=1}^{N_\varepsilon} \sup_{x \in B_x(z_i, r)} \|u_n(x)\|_{L^2(B_y(z_{i_k}, r))}^{2-\frac{2\gamma}{\alpha}} \left(\int_{B_y(z_{i_k}, r)} \frac{1}{|x-y|^\gamma} |u_n(y)|^2 dy \right)^{\frac{\gamma}{\alpha}} \\ & \leq CN_\varepsilon \|u_n(x)\|_{L^2(B_x(z_i, r))}^{\frac{\gamma}{\alpha}} \left(\sum_{i=1}^{\infty} \|u_n(x)\|_{L^2(B_x(z_i, r))}^2 \right) \left(\sup_{y \in \mathbb{R}} \int_{B(y, r)} |u_n(x)|^2 dx \right)^{1-\frac{\gamma}{\alpha}} \\ & \leq C(d+1)N_\varepsilon \|u_n\|_2^2 \|u_n(x)\|_{L^2(B_x(z_i, r))}^{\frac{\gamma}{\alpha}} \left(\sup_{y \in \mathbb{R}} \int_{B(y, r)} |u_n(x)|^2 dx \right)^{1-\frac{\gamma}{\alpha}}. \end{aligned}$$

Finally, taking n to ∞ , the second part can also be bounded by $\frac{\varepsilon}{2}$, which proves the lemma. \square

Proof of Lemma 3.2. Suppose, arguing by contradiction, that $\beta = 0$, then there exist a positive r_0 and a subsequence $\{u_{n_k}\}$ of a minimizing sequence $\{u_n\}$ such that

$$\sup_{y \in \mathbb{R}^d} \int_{B(y, r_0)} |u_{n_k}(x)|^2 dx \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Since $\{u_{n_k}\}$ is also a minimizing sequence, by Lemma 3.4, it follows that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |u_{n_k}(x)|^2 |u_{n_k}(y)|^2 dx dy \rightarrow 0, \text{ as } k \rightarrow \infty,$$

which contradicts Lemma 3.3. \square

Lemma 3.5. *Let q_1, q_2 be positive real numbers, then $E_{q_1+q_2} < E_{q_1} + E_{q_2}$.*

Proof. Given $u \in H^\alpha(\mathbb{R}^d)$ with $\|u\|_2^2 = q$, we let $u_\lambda(x) = \lambda^{\gamma_1} u(\lambda^{\gamma_2} x)$, where $\gamma_1 = \frac{2\alpha-\gamma+d}{8\alpha-2\gamma}$ and $\gamma_2 = \frac{1}{2\alpha-\gamma}$. Then $\|u_\lambda\|_2^2 = \lambda q$ and

$$E(u_\lambda) = \lambda^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}} E(u).$$

Therefore,

$$E_{\lambda q} = \inf_{\substack{u \in H^\alpha(\mathbb{R}^d) \\ M(u) = \lambda q}} E(u_\lambda) = \lambda^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}} \inf_{\substack{u \in H^\alpha(\mathbb{R}^d) \\ M(u) = q}} E(u) = \lambda^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}} E_q.$$

According to Lemma 3.1, we know that E_q is negative for all $q > 0$. For $\frac{8\alpha-2\gamma}{2\alpha-\gamma} > 1$, it follows easily from Jensen's inequality that

$$E_{q_1+q_2} = (q_1 + q_2)^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}} E_1 < (q_1^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}} + q_2^{\frac{8\alpha-2\gamma}{2\alpha-\gamma}}) E_1 = E_{q_1} + E_{q_2}.$$

□

Lemma 3.6. *Suppose $0 < \beta < q$, then $E_\beta + E_{q-\beta} \leq E_q$.*

Proof. For every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\iint_{|x-y| \geq r_\varepsilon} \frac{1}{|x-y|^\gamma} |u_n(x)|^2 |u_n(y)|^2 dx dy \leq \frac{\varepsilon}{2}, \quad (3.6)$$

and

$$\beta - \frac{\varepsilon}{4} < Q(r_\varepsilon) \leq Q(3r_\varepsilon) \leq \beta.$$

Then there exists $N_\varepsilon \in \mathbb{N}^+$ such that for every $n \geq N_\varepsilon$, we have

$$\beta - \frac{\varepsilon}{2} < Q_n(r_\varepsilon) \leq Q_n(3r_\varepsilon) < \beta + \frac{\varepsilon}{2}.$$

Next we choose $\{y_n\} \subset \mathbb{R}^d$ so that

$$\beta - \varepsilon < \int_{B(y_n, r_\varepsilon)} |u_n(x)|^2 dx \leq \int_{B(y_n, 3r_\varepsilon)} |u_n(x)|^2 dx < \beta + \varepsilon. \quad (3.7)$$

Now let us define $\phi_r(x) = \phi(\frac{x}{r})$ and $\tilde{\phi}_r(x) = \tilde{\phi}(\frac{x}{r})$, where $\phi \in C_0^\infty(B(0, 2))$ is a smooth cutoff function satisfying

$$0 \leq \phi(x) \leq 1 \text{ and } \phi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2, \end{cases} \quad (3.8)$$

and $\tilde{\phi}(x) = 1 - \phi(x)$. With this notation, we write

$$\begin{aligned} v_n(x) &= \phi_r(x - y_n) u_n(x), \\ w_n(x) &= \tilde{\phi}_r(x - y_n) u_n(x). \end{aligned} \quad (3.9)$$

It follows immediately

$$\begin{aligned} \beta - \varepsilon &< \int_{\mathbb{R}^d} |v_n(x)|^2 dx < \beta + \varepsilon, \\ q - \beta - \varepsilon &< \int_{\mathbb{R}^d} |w_n(x)|^2 dx < q - \beta + \varepsilon. \end{aligned} \quad (3.10)$$

The conclusion follows if

$$E(v_n) + E(w_n) \leq E(u_n) + c\varepsilon, \quad (3.11)$$

for some positive constant c .

To see this, note that from (3.8), there exist $\mu_n, \nu_n \in [1 - \varepsilon, 1 + \varepsilon]$ such that

$$\|\mu_n v_n\|_2^2 = \beta \text{ and } \|\mu_n w_n\|_2^2 = q - \beta.$$

we therefore deduce that

$$\begin{aligned} E_\beta &\leq E(\mu_n v_n) \leq E(v_n) + c\varepsilon, \\ E_{q-\beta} &\leq E(\mu_n w_n) \leq E(w_n) + c\varepsilon, \end{aligned}$$

for some positive constant c independent of ε . Combining the above two inequalities and using (3.11), we have

$$E_\beta + E_{q-\beta} \leq E(v_n) + E(w_n) + c\varepsilon \leq E(u_n) + c\varepsilon.$$

Passing to the limit, we can then prove the Lemma.

To sum up, what is left is to show (3.11). According to the definitions of v_n and w_n , we have

$$\begin{aligned} E(v_n) + E(w_n) &= \frac{1}{2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} v_n(x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} w_n(x)|^2 dx \\ &\quad - \frac{1}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} [|v_n(x)|^2 |v_n(y)|^2 + |w_n(x)|^2 |w_n(y)|^2] dx dy. \end{aligned}$$

Applying Lemma 2.4 and using the Sobolev's inequalities, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} [\phi_r(x - y_n) u_n(x)]|^2 dx - \int_{\mathbb{R}^d} \phi_r^2(x - y_n) |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx \\ &\leq C(\|\nabla \phi_r\|_{\frac{d}{1-\alpha}} \|(-\Delta)^{\frac{\alpha-1}{2}} u_n\|_{\frac{2d}{d-2(\alpha-1)}} + \|(-\Delta)^{\frac{\alpha}{2}} \phi_r\|_{2+\frac{d}{\alpha}} \|u_n\|_{2+\frac{4\alpha}{d}}) \\ &\leq C\left(\frac{1}{r^d} \|\nabla \phi\|_{\frac{d}{1-\alpha}} \|u_n\|_2 + \frac{1}{r^{\frac{2\alpha^2}{2\alpha+d}}} \|(-\Delta)^{\frac{\alpha}{2}} \phi\|_{2+\frac{d}{\alpha}} \|u_n\|\right). \end{aligned}$$

After taking r larger enough, we derive from the above inequality that

$$\int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} [\phi_r(x - y_n) u_n(x)]|^2 dx \leq \int_{\mathbb{R}^d} \phi_r^2(x - y_n) |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx + c\varepsilon.$$

In the same way, we see that

$$\int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} [\tilde{\phi}_r(x - y_n) u_n(x)]|^2 dx \leq \int_{\mathbb{R}^d} \tilde{\phi}_r^2(x - y_n) |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx + c\varepsilon.$$

Recalling $0 \leq \phi, \tilde{\phi} \leq 1$, we conclude from the above two inequalities that

$$\int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} v_n(x)|^2 dx + \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} w_n(x)|^2 dx \leq \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{2}} u_n(x)|^2 dx + c\varepsilon.$$

Now it remains to prove

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} [|u_n(x)|^2 |u_n(y)|^2 - |v_n(x)|^2 |v_n(y)|^2 - |w_n(x)|^2 |w_n(y)|^2] dx dy \leq c\varepsilon \quad (3.12)$$

Expanding the left hand side of (3.12) and combining the equivalent terms, we have

$$2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} (|v_n(x)|^2 |w_n(y)|^2 + 2|v_n(x)||w_n(x)||v_n(y)|^2 + 2|v_n(x)||w_n(x)||w_n(y)|^2 + 2|v_n(x)||v_n(y)||w_n(x)||w_n(y)|) dx dy \quad (3.13)$$

Indeed, except the first term $|v_n(x)|^2 |w_n(y)|^2$, the remainders are integral on the ring $B(y_n, 2r_\varepsilon) \setminus B(y_n, r_\varepsilon)$ in \mathbb{R}_x^d or \mathbb{R}_y^d (or both). Therefore, from (3.7), we have

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} |v_n(x)||w_n(x)||v_n(y)|^2 dx dy \\ & \leq \sup_{x \in \mathbb{R}^d} \int_{B_y(y_n, 2r)} \frac{1}{|x-y|^\gamma} |v_n(y)|^2 dy \int_{B_x(y_n, 2r+r_\varepsilon) \setminus B_x(y_n, r)} |v_n(x)||w_n(x)| dy \quad (3.14) \\ & \leq \|u_n\| \int_{B(y_n, 2r_\varepsilon) \setminus B(y_n, r_\varepsilon)} |u_n(x)|^2 dx \leq c\varepsilon \end{aligned}$$

Similarly,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} (|v_n(x)||w_n(x)||w_n(y)|^2 + |v_n(x)||v_n(y)||w_n(x)||w_n(y)|) dx dy \leq c\varepsilon \quad (3.15)$$

To estimate the first term, recalling (3.6), we only need to deal with the integral on the set $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x-y| \leq r_\varepsilon\}$. Similar arguments as above imply that

$$\begin{aligned} & \iint_{|x-y| \leq r_\varepsilon} \frac{1}{|x-y|^\gamma} |v_n(x)|^2 |w_n(y)|^2 dx dy \\ & \leq \sup_{y \in \mathbb{R}^d} \int_{B_x(y_n, 2r_\varepsilon)} \frac{1}{|x-y|^\gamma} |v_n(x)|^2 dx \int_{B_x(y_n, 3r_\varepsilon) \setminus B_x(y_n, r_\varepsilon)} |w_n(y)|^2 dy \quad (3.16) \\ & \leq \|u_n\| \int_{B(y_n, 3r_\varepsilon) \setminus B(y_n, r_\varepsilon)} |u_n(x)|^2 dx \leq c\varepsilon \end{aligned}$$

From (3.13)-(3.16), we proved (3.12). This finishes the proof. \square

Proof of Theorem 1.2. Recalling the definition of β in (3.1), by Lemma 3.2, Lemma 3.5 and Lemma 3.6, we know that every minimizing sequence $\{u_n\}$ for problem (1.5) has a subsequence, denoted again by $\{u_n\}$, satisfying

$$\lim_{r \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^d} \int_{B(y, r)} |u_n(x)|^2 dx = \beta = q, \quad (3.17)$$

which implies that for every positive $\varepsilon > 0$, there exist $r_\varepsilon > 0$, $n_\varepsilon \in \mathbb{N}^+$ and $\{y_n\} \subset \mathbb{R}^d$ such that for each $n > n_\varepsilon$ and $r > r_\varepsilon$,

$$\int_{B(y_n, r)} |u_n(x)|^2 dx > q - \varepsilon. \quad (3.18)$$

According to Lemma 3.3, $\{u_n(\cdot - y_n)\}$ is bounded in $H^\alpha(\mathbb{R}^d)$. By choosing subsequence if necessary, there exists $g \in H^\alpha(\mathbb{R}^d)$ such that,

$$u_n(\cdot - y_n) \rightharpoonup g \text{ weakly in } H^\alpha(\mathbb{R}^d).$$

We can find $R_\varepsilon > r_\varepsilon$ such that $\|g\|_{L^2(\mathbb{R}^d \setminus B(0, R_\varepsilon))} < \frac{\varepsilon}{2}$. Furthermore, by Lemma 2.5, there exists $N_\varepsilon \in \mathbb{N}^+$ with $N_\varepsilon > n_\varepsilon$ such that for $n > N_\varepsilon$, we have

$$\|u_n(\cdot - y_n) - g\|_{L^2(B(0, R_\varepsilon))} < \frac{\varepsilon}{2}.$$

It follows immediately from the above that

$$\begin{aligned} \|g\|_2 &\geq \|u_n\|_2 - \|u_n(\cdot - y_n) - g\|_{L^2(B(0, r_\varepsilon))} - \|u_n(\cdot - y_n) - g\|_{L^2(\mathbb{R}^d \setminus B(0, r_\varepsilon))} \\ &\geq \|u_n\|_{L^2(B(y_n, r_\varepsilon))} - \|u_n(\cdot - y_n) - g\|_{L^2(B(0, r_\varepsilon))} - \|g\|_{L^2(\mathbb{R}^d \setminus B(0, r_\varepsilon))} \\ &\geq \sqrt{q - \varepsilon} - \varepsilon, \end{aligned} \quad (3.19)$$

which implies, by passing to the limit, $\|g\|_2^2 \geq q$. On the other hand, the weak lower semi-continuous deduces

$$q \leq \|g\|_2^2 \leq \liminf_{n \rightarrow +\infty} \|u_n\|_2^2 = q. \quad (3.20)$$

Therefore, $\|g\|_2^2 = q$, and consequently,

$$u_n(\cdot - y_n) \rightarrow g \text{ strongly in } L^2(\mathbb{R}^d),$$

since $\{u_n(\cdot - y_n)\}$ converges weakly in $H^\alpha(\mathbb{R}^d)$. Moreover, we have

$$\begin{aligned} &\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} (|u_n(x - y_n)|^2 |u_n(y - y_n)|^2 - |g(x)|^2 |g(y)|^2) dx dy \\ &\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} [|u_n(x - y_n)|^2 (|u_n(y - y_n)|^2 - |g(y)|^2) \\ &\quad + |g(y)|^2 (|u_n(x - y_n)|^2 - |g(x)|^2)] dx dy \\ &\leq C(\|u_n\| + \|g\|) \| |u_n(\cdot - y_n)|^2 - |g|^2 \|_2 \\ &\leq C(\|u_n\| + \|g\|) (\|u_n\|_2 + \|g\|_2) \|u_n(\cdot - y_n) - g\|_2 \rightarrow 0, \end{aligned} \quad (3.21)$$

as $n \rightarrow \infty$. Applying the weak lower semi-continuous again, we deduce that

$$\|g\|_{\dot{H}^\alpha(\mathbb{R}^d)} \leq \liminf_{n \rightarrow +\infty} \|u_n\|_{\dot{H}^\alpha(\mathbb{R}^d)}. \quad (3.22)$$

From (3.21) and (3.22), it follows immediately that

$$\beta \leq E(g) \leq \liminf_{n \rightarrow +\infty} E(u_n) = \beta. \quad (3.23)$$

Hence, g is a minimizer of problem (1.5) and

$$u_n(\cdot - y_n) \rightarrow g \text{ in } H^\alpha(\mathbb{R}^d). \quad (3.24)$$

Next, arguing by contradiction, we prove (1.10). Assume that there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\inf_{g \in G_q} \|u_{n_k} - g\| \geq \varepsilon_0 > 0. \quad (3.25)$$

From what has already been proved, we know that there exist a subsequence of $\{u_{n_k}\}$, denoted again by $\{u_{n_k}\}$, and $\{y_{n_k}\} \in \mathbb{R}^d$ such that

$$u_{n_k}(\cdot - y_{n_k}) \rightharpoonup g \text{ in } H^\alpha(\mathbb{R}^d).$$

Since $g(\cdot + y_{n_k}) \in G_q$, it follows that

$$\|u_{n_k} - g(\cdot + y_{n_k})\| = \|u_{n_k}(\cdot - y_{n_k}) - g\| \rightarrow 0,$$

which contradicts (3.25). \square

Proof of Theorem 1.3. Consider a minimizer $g \in G_q$. Assume that $g \in L^p(\mathbb{R}^d)$, then $(|\cdot|^{-\gamma} * |g|^2)g \in L^p(\mathbb{R}^d)$. Note that equation (1.4) can be written in the form

$$\mathcal{F}^{-1}[(1 + |\xi|^{2\alpha})\mathcal{F}g] = (|\cdot|^{-\gamma} * |g|^2)g. \quad (3.26)$$

It follows that $g \in H^{2\alpha,p}(\mathbb{R}^d)$. By Sobolev's embedding theorem, we have

$$g \in L^q(\mathbb{R}^d) \text{ for all } \frac{1}{q} \in \left[\frac{1}{p} - \frac{2\alpha}{d}, \frac{1}{p} \right]. \quad (3.27)$$

Consider the sequence $\{q_i\}$ defined by

$$q_0 = 2 \text{ and } q_{i+1} = \frac{dq_i}{d - 2\alpha q_i} \text{ for } i \in \mathbb{N}^+$$

Since

$$\frac{1}{q_{i+1}} - \frac{1}{q_i} = -\frac{2\alpha}{d} < 0,$$

we deduce that $\frac{1}{q_i} \rightarrow -\infty$ as $i \rightarrow +\infty$, so there exist $i_0 \in \mathbb{N}^+$ such that

$$\frac{1}{q_i} > 0 \text{ for } 0 \leq i \leq i_0 \text{ and } \frac{1}{q_{i_0+1}} \leq 0.$$

By an induction argument and (3.27), it is not difficult to show that $g \in L^{q_{i_0}}(\mathbb{R}^d)$. Applying once again (3.27), we deduce that

$$g \in L^q(\mathbb{R}^d) \text{ for all } \frac{1}{q} \in \left[\frac{1}{q_{i_0+1}}, \frac{1}{q_{i_0}} \right]. \quad (3.28)$$

In particular, we can take $q = \infty$, so that $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We obtain immediately that $(|\cdot|^{-\gamma} * |g|^2)g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Using (3.26) again, we have $g \in H_p^{2\alpha}(\mathbb{R}^d)$ for all $p \in [2, \infty)$. By Sobolev's embedding, $g \in C^{[2\alpha], 2\alpha - [2\alpha]}(\mathbb{R}^d)$.

We now turn to the part (2) of Theorem 1.3. Arguing by contradiction, we assume that $|g| \notin S'$. Then, Lemma 2.6 gives us

$$\|g^*\|_{\dot{H}^\alpha(\mathbb{R}^d)} \leq \|g\|_{\dot{H}^\alpha(\mathbb{R}^d)}. \quad (3.29)$$

Since $\frac{1}{|x|^{-\gamma}} \in S$ and satisfies (2.7), it follows immediately from Lemma 2.8 that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} |g(x)|^2 |g(y)|^2 dx dy < \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} |g^*(x)|^2 |g^*(y)|^2 dx dy, \quad (3.30)$$

unless $|g| \in S'$ for some $y \in \mathbb{R}^d$. Recalling (2.4) and combining (3.29) and (3.30), we conclude that

$$\|g^*\|_2 = \|g\|_2 = q \text{ and } E(g^*) < E(g) = E_q,$$

which contradicts the definition of E_q . Hence, we have proved $|g| \in S'$. Furthermore $|g| \in G_q$, since $E(|g|) = E(g^*) = q$.

Next we claim that $|g|(x) > 0$ for all $x \in \mathbb{R}^d$. To this end, we arguing by contradiction. Suppose that there exists $x_0 \in \mathbb{R}^d$ such that $|g|(x_0) = 0$. Then, it follows from the equation (1.4) that $(-\Delta)^\alpha |g|(x_0) = 0$. By Lemma 2.2, we have

$$(-\Delta)^\alpha |g|(x_0) = C_{d,\alpha} \left(\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon \leq |x_0 - y| \leq r} \frac{-|g|(y)}{|x_0 - y|^{d+2\alpha}} dy + \int_{\mathbb{R}^d \setminus B(x_0, r)} \frac{-|g|(y)}{|x_0 - y|^{d+2\alpha}} dy \right) = 0,$$

which implies

$$\int_{\mathbb{R}^d \setminus B(x_0, r)} \frac{|g|(y)}{|x_0 - y|^{d+2\alpha}} dy = 0 \text{ for all } r > 0,$$

Therefore, $|g| \equiv 0$ on \mathbb{R}^d , which contradicts $|g| \in G_q$. Let $u = \Re g$, $v = \Im g$, we have $g = u + iv$. It follows that

$$(-\Delta)^\alpha u - (|\cdot|^{-\gamma} * |g|^2)u = \omega u \quad (3.31)$$

$$(-\Delta)^\alpha v - (|\cdot|^{-\gamma} * |g|^2)v = \omega v, \quad (3.32)$$

Repeating the similar argument as before for the linear equation

$$(-\Delta)^\alpha h - (|\cdot|^{-\gamma} * |g|^2)h = \omega h, \quad (3.33)$$

we know that u, v are continuous and $|u|, |v| > 0$. Therefore, u and v both have constant signs. We claim that there exists constants α, β such that $u = \alpha|g|$ and $v = \beta|g|$. If this were not the case, there would exist a constant c such that $w = u - c|g|$ takes both positive and negative values. It is easy to see that w also satisfies the equation (3.33), which is a contradiction. Likewise, the same conclusion is true for v . Thus, we have proved that $g = \alpha|g| + i\beta|g| = e^{i\theta}|g|$, where θ is a constant satisfying $\tan(\theta) = \frac{\beta}{\alpha}$. Thus we have finished our proof of Theorem 1.3. \square

Remark 3.7. Actually we can prove the existence of radial standing waves of the equation (1.1) in a much simple way by symmetric decreasing rearrangements of minimizing sequence in Theorem 1.2. In this way, however, we may not exclude the possibility of nonradial standing waves of the equation (1.1) and could not deduce (1.10) in Theorem 1.2, which plays a key role in the proof of Theorem 1.4.

Proof of Theorem 1.4. We arguing by way of contradiction. Suppose that the set G_q is not $H^\alpha(\mathbb{R}^d)$ -stable. Then there exist $\varepsilon_0 > 0$, a sequence $\{u_m^{(0)}\}$ in $H^\alpha(\mathbb{R}^d)$ and $t_m \in \mathbb{R}$ such that

$$\inf_{g \in G_q} \|u_m^{(0)} - g\| < \frac{1}{m}, \quad (3.34)$$

and

$$\inf_{g \in G_q} \|u_m(t_m) - g\| \geq \varepsilon_0, \quad (3.35)$$

where $u_m \in C(\mathbb{R}, H^\alpha(\mathbb{R}^d))$ are solutions to NLS (1.1) with initial data $u_m(x, 0) = u_m^{(0)}(x)$. From (3.34), we have as $m \rightarrow +\infty$

$$\|u_m^{(0)}\|_2^2 \rightarrow q, \quad E(u_m^{(0)}) \rightarrow E_q.$$

Hence, we can find $\{\mu_m\} \subset \mathbb{R}$, satisfying $\|\mu_m u_m^{(0)}\|_2^2 = q$ and $\mu_m \rightarrow 1$, such that $\{\mu_m u_m^{(0)}\}$ is a minimizing sequence for the problem (1.5). By the conservation laws,

$$\begin{aligned} \|\mu_m u_m(t_m)\|_2^2 &= \|\mu_m u_m^{(0)}\|_2^2 = q, \\ E(\mu_m u_m(t_m)) &\rightarrow E_q, \end{aligned}$$

we know that $\{\mu_m u_m(t_m)\}$ is also a minimizing sequence for the problem (1.5). According to Theorem 1.2, there exist a subsequence $\{\mu_{m_k} u_{m_k}(t_{m_k})\}$ of $\{\mu_m u_m(t_m)\}$, and $\{g_{m_k}\}$ in G_q such that

$$\|\mu_{m_k} u_{m_k}(t_{m_k}) - g_{m_k}\| < \frac{\varepsilon_0}{2}, \quad (3.36)$$

for sufficiently large m_k . It follows that

$$\begin{aligned} \varepsilon &\leq \|u_{m_k}(t_{m_k}) - g_{m_k}\| \leq \|u_{m_k}(t_{m_k}) - \mu_{m_k} u_{m_k}(t_{m_k})\| + \|\mu_{m_k} u_{m_k}(t_{m_k}) - g_{m_k}\| \\ &\leq |\mu_{m_k} - 1| \|u_{m_k}(t_{m_k})\| + \frac{\varepsilon_0}{2}, \end{aligned}$$

which leads to a contradiction since $\mu_m \rightarrow 1$. Therefore, the set G_q is $H^\alpha(\mathbb{R}^d)$ -stable with respect to NLS (1.1). \square

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Prof. Daomin Cao and Prof. Pigong Han for several helpful comments and for many stimulating conversations.

REFERENCES

- [1] X. CABRE AND Y. SIRE, *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, arXiv preprint arXiv:1012.0867, (2010).
- [2] T. CAZENAVE, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics 10, 2003.
- [3] T. CAZENAVE AND P. LIONS, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Communications in Mathematical Physics, 85 (1982), 549–561.
- [4] Y. CHO, G. HWANG, H. HAJAIEJ, AND T. OZAWA, *On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity*, arXiv preprint arXiv:1209.5899, (2012).
- [5] Y. CHO, G. HWANG, S. KWON, AND S. LEE, *On the finite time blowup for mass-critical Hartree equations*, arXiv preprint arXiv:1208.2302, (2012).
- [6] ———, *Profile decompositions and blowup phenomena of mass critical fractional Schrödinger equations*, arXiv preprint arXiv:1208.2303, (2012).
- [7] R. FRANK AND E. LENZMANN, *On ground states for the L^2 -critical boson star equation*, arXiv preprint arXiv:0910.2721, (2009).
- [8] Q. GUAN AND Z. MA, *Reflected symmetric α -stable processes and regional fractional Laplacian*, Probability theory and related fields, 134 (2006), 649–694.
- [9] B. GUO, Y. HAN, AND J. XIN, *Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation*, Applied Mathematics and Computation, 204 (2008), 468–477.

- [10] B. GUO AND D. HUANG, *Existence and stability of standing waves for nonlinear fractional Schrödinger equations*, Journal of Mathematical Physics, 53 (2012), 083702–083702.
- [11] B. GUO AND Z. HUO, *Global well-posedness for the fractional nonlinear Schrödinger equation*, Communications in Partial Differential Equations, 36 (2010), 247–255.
- [12] H. HAJAIEJ, *Some fractional functional inequalities and applications to some constrained minimization problems involving a local non-linearity*, arXiv preprint arXiv:1104.1414, (2011).
- [13] T. KATO AND G. PONCE, *Commutator estimates and the Euler and Navier-Stokes equations*, Communications on Pure and Applied Mathematics, 41 (1988), 891–907.
- [14] N. LASKIN, *Fractional quantum mechanics*, Physical Review E, 62 (2000), 3135.
- [15] ———, *Fractional quantum mechanics and Levy path integrals*, Physics Letters A, 268 (2000), 298–305.
- [16] ———, *Fractional Schrödinger equation*, Physical Review E, 66 (2002), 056108.
- [17] E. LIEB, *Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation*, Studies in Applied Mathematics, 57 (1977), 93–105.
- [18] E. LIEB AND M. LOSS, *Analysis*, American Mathematical Society, Providence, RI, 2001.
- [19] P. LIONS, *The concentration-compactness principle in the calculus of variations. the locally compact case. i*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 1 (1984), 109–145.
- [20] ———, *The concentration-compactness principle in the calculus of variations. the locally compact case. ii*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 1 (1984), 223–283.
- [21] ———, *Solutions of Hartree-Fock equations for Coulomb systems*, Communications in Mathematical Physics, 109 (1987), 33–97.
- [22] F. RIESZ, *Sur une inegalite integrale*, Journal of the London Mathematical Society, 1 (1930), 162.
- [23] E. VALDINOCI, *From the long jump random walk to the fractional Laplacian*, Boletín de la Sociedad Española de Matemática Aplicada, (2009), 33–44.

DAN WU

ACADEMY OF MATHEMATICS AND SYSTEM SCIENCES, BEIJING 100190, CHINA.

E-mail address: danwu@amss.ac.cn